Phys 410
Spring 2013
Lecture #4 Summary
30 January, 2013

We considered vertical motion of a point-like object subjected to a linear drag force. Newton's second law of motion for the vertical component of motion is given by $m\dot{v}_y = mg - bv_y$, with y-positive in the downward direction. Starting from rest, the particle will initially experience no drag force. As it accelerates downward, the drag force will increase steadily. At some point the drag force will equal the force of gravity, and the acceleration will cease. The object will be moving at the terminal velocity from that time onward, given by the balance of the two forces as $v_{ter} = \frac{gm}{b} = \tau g$. The equation of motion can be written as $m\dot{v}_y = b\left(v_y - v_{ter}\right)$, with a solution $v_y(t) = v_{y0}e^{-t/\tau} + v_{ter}\left(1 - e^{-t/\tau}\right)$. This equation shows that the velocity starts as v_{y0} and then switches over to v_{ter} as time evolves. The time scale for this crossover is $\tau = m/b$, the characteristic time governing the horizontal motion as well. The velocity equation can be integrated to find the vertical position as a function of time:
$y(t) - y(0) = v_{ter}t \quad - \quad \left(v_{y0} - v_{ter}\right)\tau \left(1 - e^{-t/\tau}\right)$.

Combining the solutions for the x-motion and y-motion, we can now construct a solution for general motion in the xy-plane for a particle experiencing linear drag. Solving the $x(t)$ equation from the last lecture for t in terms of x, and putting this into the $y(t)$ equation above, we can find the trajectory of the particle: $y(x) = \frac{v_{y0} + v_{ter}}{v_{x0}}x + \ v_{ter}\tau \ \ln\left(1 - \frac{x}{v_{x0}\,\tau}\right)$. The trajectory in the absence of drag is given by $y(x) = \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2$. The main difference is in the second term. The logarithm has a negative divergence as the argument goes to zero. This occurs when $x = \tau\ v_{x0}$, showing that the horizontal range of the projectile is limited by this value.

We also considered motion with quadratic drag, $f = -c\ v^2$. The equation of motion is $m\dot{\vec{v}} = m\vec{g} - c\ v^2\hat{v}$. Note that the last term can also be written as $-c\ v\ \vec{v}$, allowing a decomposition into two scalar differential equations: $m\dot{v}_x = 0 - c\ \sqrt{v_x^2 + v_y^2}\ v_x$ and $m\dot{v}_y = mg - c\ \sqrt{v_x^2 + v_y^2}\ v_y$. Note that these equations do not separate cleanly into a v_x-only and a v_y-only set of equations, as they did in the linear drag case. In this case there is no analytical general solution for this pair of equations. We will consider motion exclusively in the x-direction to simplify the problem.

If we confine the particle to move only in the the x-direction ($v_y = 0$), the first equation reduces to $m\dot{v}_x = -c\ v_x^2$, with solution $v_x = \frac{v_0}{1+t/\tau}$, where we have defined a new characteristic time scale $\tau \equiv m/v_0 c$, and v_0 is the initial velocity. Here we see that the velocity relaxes more slowly than in the linear drag case, where the relaxation was exponential rather than algebraic. The velocity equation can be integrated to find the position of the particle along the x-axis: $x(t) - x(0) = v_0 \tau \ln(1 + t/\tau)$. In this case the particle continues to move forever in the x-direction as time increases. Solving the equation for the vertical-only motion is left as a homework problem.